Fast Fourier Transform (FFT)

Lets calculate the DFT of a sequence with N=4 and k=1

$$X(1) = \sum_{n=0}^{3} x(n)W_4^n$$

$$= x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2 + x(3)W_4^3$$

For a single value of k we have

4 → Multiplication

3 → Addition

For example N=1024

DFT method

$$Complex\ multiplication = N^2$$
$$= 1024^2$$
$$= 1048576$$

$$Complex\ Addition = N(N-1)$$
$$= 1024(1024-1)$$
$$= 1047552$$

Fast Fourier Transform (FFT)

- *DFT takes more time and resources*
- *Not much efficient*
- *Much complex*

- *So we come into a new algorithm to make the calculations fast known as fast Fourier Transform (FFT)*
- *It is a highly efficient procedure for computing the DFT of a sequence for computing the DFT of a finite sequence and require less number of computation than that of direct evaluation of DFT*
- *FFT is based on decomposition and breaking the transform into smaller transform and combine them to get total transform*
- *FFT make use of the symmetry and periodicity property of twiddle factor*

FFT method

For example N=1024

$$\text{Complex multiplication} = \frac{N}{2}\log_2 N$$

$$= \frac{1024}{2}\log_2 1024$$

$$= 5120$$

$$\text{Complex Addition} = N\log_2 N$$

$$= 1024\log_2 1024$$

$$= 1024$$

Fast Fourier Transform (FFT)

Let us recollect the twiddle factor

DFT

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad ,0 \le \mathrm{k} \le \mathrm{N}-1}$$

$$W_N = e^{-\frac{j2\pi}{N}}$$

$$W_N^k = e^{\left(-\frac{j2\pi}{N}\right)k}$$

For N=4 & k=0

$$\boxed{W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1}$$

For N=4 & k=1

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).1} = e^{-\frac{j\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$\boxed{W_4^1 = -j}$$

$$e^{-j\theta} = \cos(\theta) - jsin(\theta)$$

For N=8 & k=0

$$\boxed{W_8^0 = e^{\left(-\frac{j2\pi}{8}\right)0} = 1}$$

For N=8 & k=1

$$W_8^1 = e^{\left(-\frac{j2\pi}{8}\right).1} = e^{\frac{-j\pi}{4}}$$

$$= \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right)$$

$$\boxed{W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.7071 - j0.7071}$$

For N=8 & k=2

$$W_8^2 = e^{\left(-\frac{j2\pi}{8}\right).2} = e^{\frac{-j\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$\boxed{W_8^2 = -j}$$

For N=8 & k=3

$$W_8^3 = e^{\left(-\frac{j2\pi}{8}\right).3} = e^{\frac{-j3\pi}{4}}$$

$$= \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right)$$

$$W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^3 = -0.7071 - j0.7071$$

Fast Fourier Transform (FFT)

Decimation in Time (DIT)

- Also known as Radix DIT FFT algorithm
- The number of output points N can be expressed as a power of 2 ($N=2^M$)

Let x(n) is an N-point sequence and we are dividing it into two (even $x_e(n)$ & odd $x_o(n)$) parts

$$x_e(n) = x(2n) \qquad x_o(n) = x(2n+1)$$

We know DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk}$$

$$W_N^2 = e^{\left(-\frac{j2\pi}{N}\right)2} = e^{\left(-\frac{j2\pi}{\frac{N}{2}}\right)} = W_{\frac{N}{2}}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_{\frac{N}{2}}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_{\frac{N}{2}}^{nk}$$

$$\boxed{X(k) = X_e(k) + W_N^k X_o(k) \qquad \textit{For } k < N/2}$$

From symmetry property : $W_N^{k+\frac{N}{2}} = -W_N^k$

Then

$$\boxed{X(k) = X_e\left(k - \frac{N}{2}\right) - W_N^{k-\frac{N}{2}} X_o\left(k - \frac{N}{2}\right) \qquad \textit{For } k > N/2}$$

Decimation in Time (DIT)

$$X(k) = X_e(k) + W_N^k X_o(k)$$

$$X(k) = X_e\left(k - \frac{N}{2}\right) - W_N^{k-\frac{N}{2}} X_o\left(k - \frac{N}{2}\right)$$

Example : For N = 8

even	odd
$x_e(0) = x(0)$	$x_o(0) = x(1)$
$x_e(1) = x(2)$	$x_0(1) = x(3)$
$x_e(2) = x(4)$	$x_o(2) = x(5)$
$x_e(3) = x(6)$	$x_o(3) = x(7)$

$$X(k) = x_e(k) + W_8^k x_o(k), \qquad for\ 0 \le k \le 3$$

$$X(k) = x_e(k-4) - W_8^{k-4} x_o(k-4), \qquad for\ 4 \le k \le 7$$

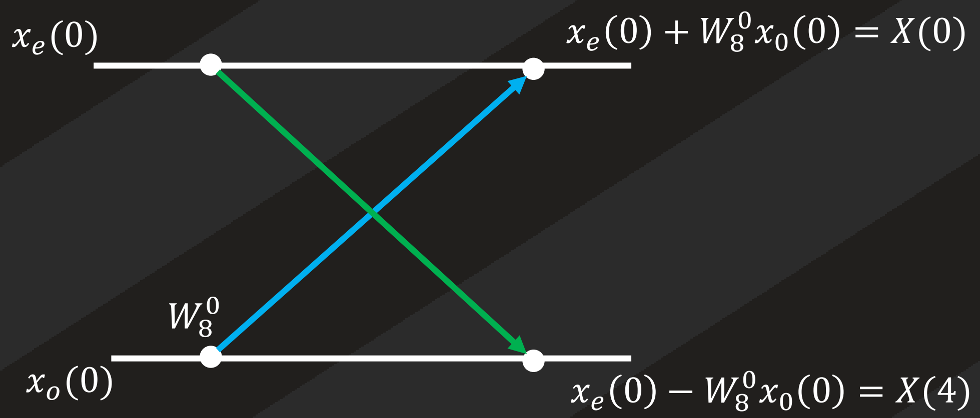
$$X(0) = x_e(0) + W_8^0 x_o(0) \qquad X(4) = x_e(0) - W_8^0 x_o(0)$$

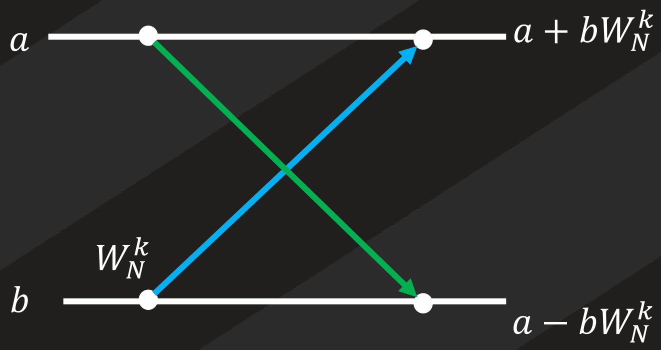
$$X(1) = x_e(1) + W_8^1 x_o(1) \qquad X(5) = x_e(1) - W_8^1 x_o(1)$$

$$X(2) = x_e(2) + W_8^2 x_o(2) \qquad X(6) = x_e(2) - W_8^2 x_o(2)$$

$$X(3) = x_e(3) + W_8^3 x_o(3) \qquad X(7) = x_e(3) - W_8^3 x_o(3)$$

This operation can be represented by a butterfly diagram





Decimation in Time (DIT)

Steps to follow

Step 1 : Find the number of input samples (N)

Step 2 : Bir reversal

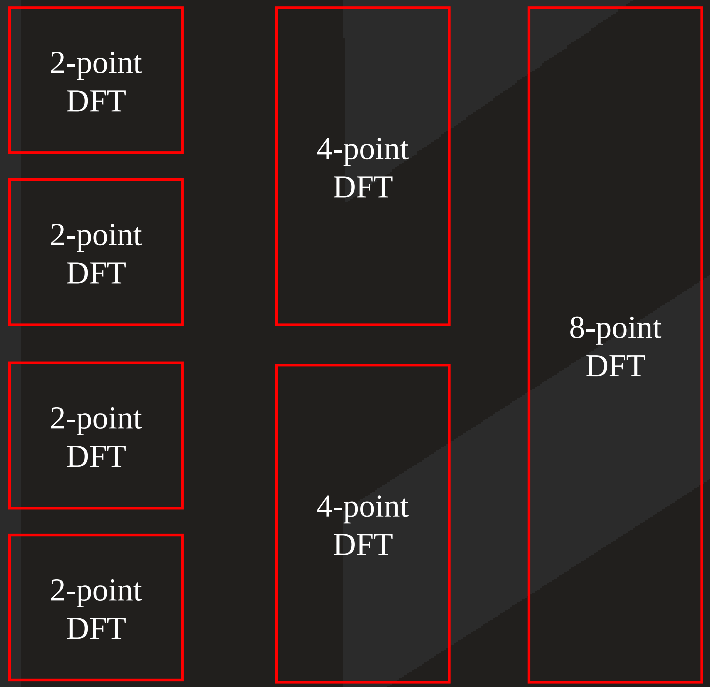
Step 3 : Calculate the number of stages ($M = \log_2 N$)

Step 4 : Calculate the number of max butterflies in stage (N/2)

Step 5 : Calculate the twiddle factor

Step 6 : Evaluate the N point DFT using butterfly diagram

Step7 : The DFT output is in normal order



Input	Binary	Bit-reversed	Revised samples
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Decimation in Time (DIT)

Q) Find the DFT of a sequence x(n) = {0, 1, 2, 3} using DIT algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : Bit reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

Decimation in Time (DIT)

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^M} \qquad t = 0, 1, 2, \dots 2^{M-1} - 1$$

Stage =1 $\quad (M = 1)$

$t = 0$

$$k = \frac{Nt}{2^M} \quad = \frac{4.0}{2^1} \quad = 0$$

W_4^0

$$\boxed{W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1}$$

Stage =2 $\quad (M = 2)$

$t = 0,1$

for t=0

$$k = \frac{4.0}{2^2} = 0$$

for t=1

$$k = \frac{4.1}{2^2} = 1$$

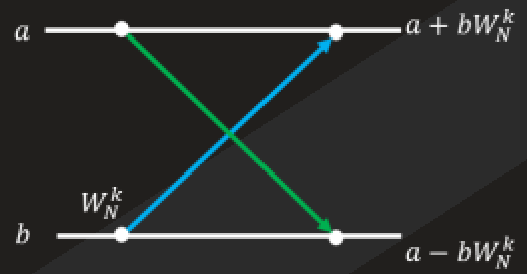
$W_4^0 \qquad W_4^1$

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).1} \quad = e^{-\frac{j\pi}{2}} \quad = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$\boxed{W_4^1 = -j}$$

Decimation in Time (DIT)

Step 6 : Evaluate the N point DFT using butterfly diagram

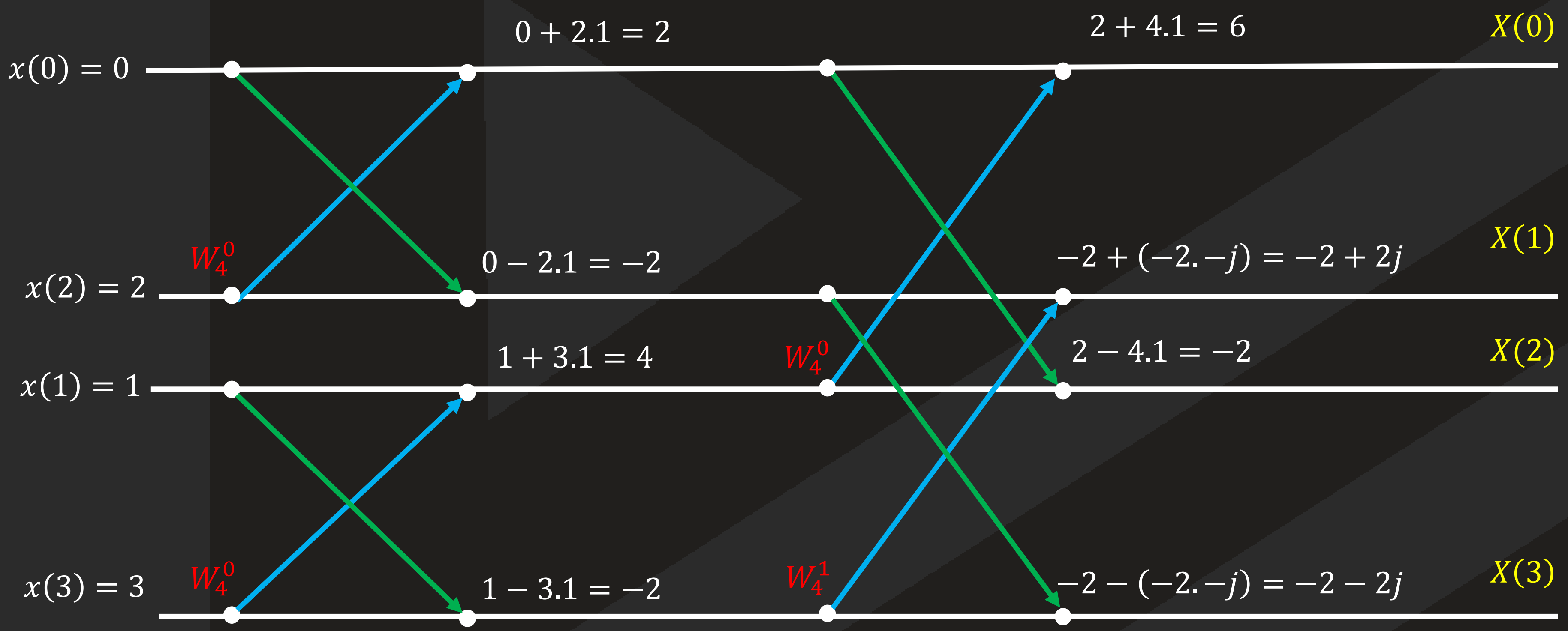


$x(n) = \{0, 1, 2, 3\}$

$W_4^0 = 1 \quad W_4^1 = -j$

Stage 1

Stage 2



Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step7 : The DFT output is in normal order

$X(k) = \{6, -2 + 2j, -2 \ \ - 2 - 2j\}$

Decimation in Time (DIT)

Q) Find the DFT of a sequence x(n) = {1, 2, 3, 4, 4, 3, 2, 1 } using DIT algorithm

Solution

Step 1 : Find the number of input samples (N)

N=8

Step 2 : Bit reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Step 3 : Calculate the number of stages ($M = \log_2 N$)

$$M = \log_2 N \quad = \log_2 8$$

$$M = 3$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{8}{2} = 4$$

Decimation in Time (DIT)

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^M} \qquad t = 0, 1, 2, \ldots 2^{M-1} - 1$$

Stage =1

$(M = 1)$

$t = 0$

$$k = \frac{Nt}{2^M} = \frac{8.0}{2^1} = 0$$

W_8^0

$$\boxed{W_8^0 = e^{\left(-\frac{j2\pi}{8}\right)0} = 1}$$

Stage =2

$(M = 2)$

$t = 0,1$

for t=0

$$k = \frac{8.0}{2^2} = 0 \qquad W_8^0$$

for t=1

$$k = \frac{8.1}{2^2} = 2 \qquad W_8^2$$

$$\boxed{W_8^2 = -j}$$

$$W_8^3 = e^{\left(-\frac{j2\pi}{8}\right).3} = e^{\frac{-j3\pi}{4}}$$

$$= \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right)$$

$$\boxed{\begin{gathered} W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ W_8^3 = -0.7071 - j0.7071 \end{gathered}}$$

Stage =3

$(M = 3)$

$t = 0,1,2,3$

for t=0

$$k = \frac{8.0}{2^3} = 0 \qquad W_8^0$$

$$\boxed{W_8^0 = 1}$$

for t=1

$$k = \frac{8.1}{2^3} = 1 \qquad W_8^1$$

$$\boxed{W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}$$

for t=2

$$k = \frac{8.2}{2^3} = 2 \qquad W_8^2$$

$$\boxed{W_8^2 = -j}$$

for t=3

$$k = \frac{8.3}{2^3} = 3 \qquad W_8^3$$

$$\boxed{W_8^2 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}$$

$$W_8^2 = e^{\left(-\frac{j2\pi}{8}\right).2}$$

$$= e^{\frac{-j\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$\boxed{W_8^2 = -j}$$

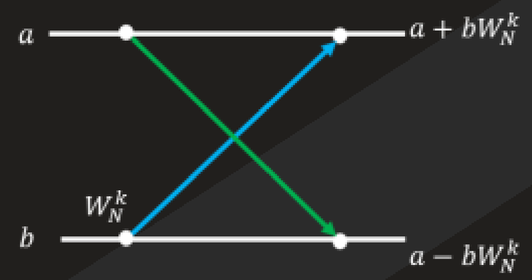
$$W_8^1 = e^{\left(-\frac{j2\pi}{8}\right).1} = e^{\frac{-j\pi}{4}}$$

$$= \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right)$$

$$\boxed{W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.7071 - j0.7071}$$

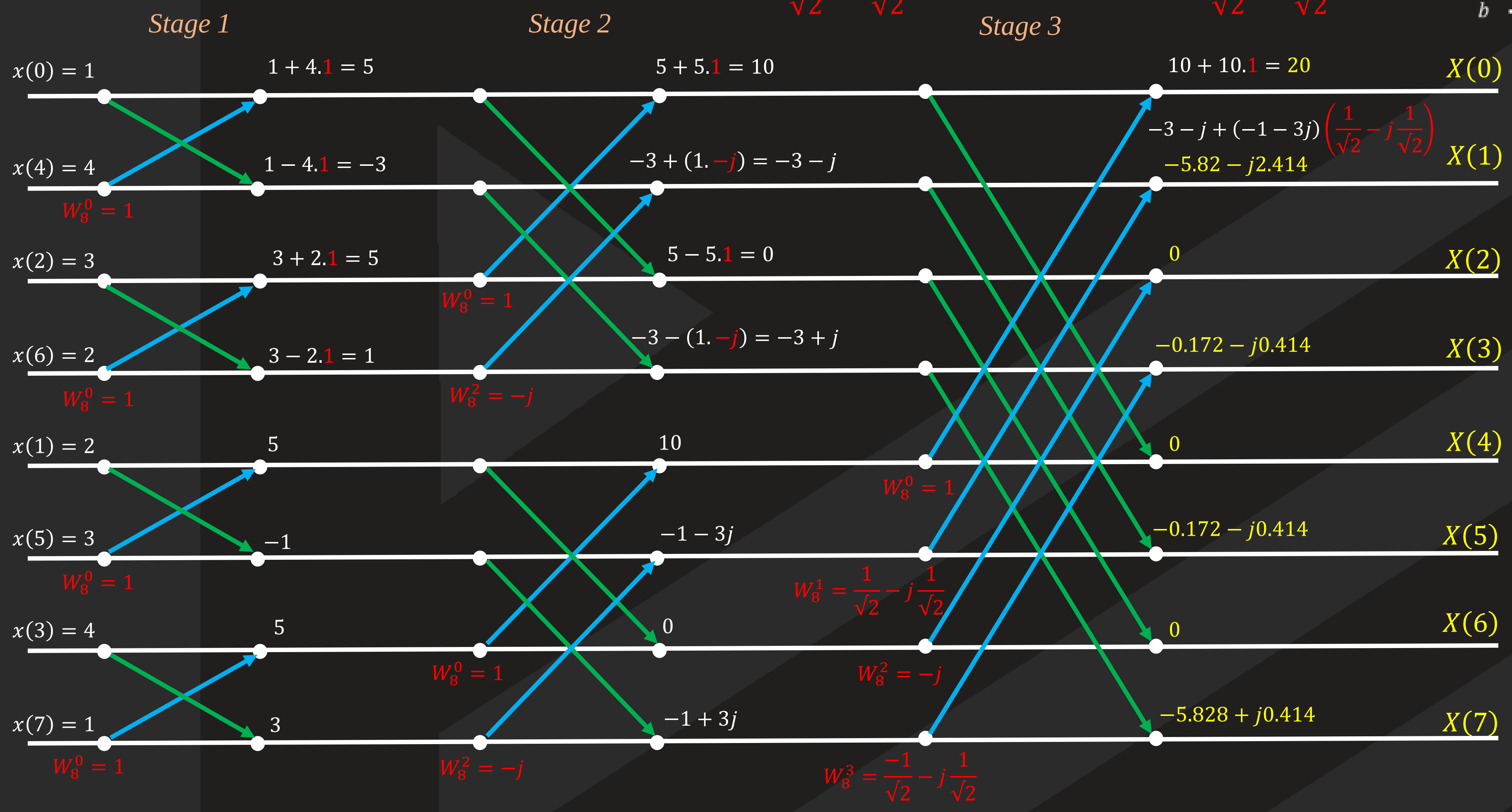
Decimation in Time (DIT)

Step 6 : Evaluate the N point DFT using butterfly diagram



$x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

$W_8^0 = 1$ $\quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$ $\quad W_8^2 = -j$ $\quad W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$



Input	Revised samples
x(0)	x(0)
x(1)	x(4)
x(2)	x(2)
x(3)	x(6)
x(4)	x(1)
x(5)	x(5)
x(6)	x(3)
x(7)	x(7)

Step7 : The DFT output is in normal order

$X(k) = \{20, -5.82 - j2.414, 0, -0.172 - j0.414, 0, -0.172 - j0.414, 0, -5.828 + j0.414\}$

Fast Fourier Transform (FFT)

$W_N^{\frac{Nk}{2}} = e^{\left(\frac{-j2\pi}{N}\right)\frac{Nk}{2}} = e^{-j\pi k}$

Decimation in Frequency (DIF)

- Based on the decomposition of the DFT computation by forming smaller and smaller sub sequences
- In DIF the output sequence X(k) is divided into smaller and smaller sub sequences

Let x(n) is an N-point sequence and we are dividing it into two parts

$x_1(n) = x(n)$ $\quad$ $x_2(n) = x\left(n + \frac{N}{2}\right)$

$n = 0,1,2,\ldots\frac{N}{2} - 1$ $\quad$ $n = 0,1,2,\ldots\frac{N}{2} - 1$

We know DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n)W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n)W_N^{\left(n+\frac{N}{2}\right)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n)W_N^{nk} + W_N^{\frac{NK}{2}} \sum_{n=0}^{\frac{N}{2}-1} x_2(n)W_N^{nk}$$

When k is even $e^{-j\pi k} = 1$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} x_1(n)W_N^{nk} + e^{-j\pi k}\sum_{n=0}^{\frac{N}{2}-1} x_2(n)W_N^{nk} = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) + x_2(n)\right)W_N^{2nk}$$

$$\boxed{X(2k) = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) + x_2(n)\right)W_{\frac{N}{2}}^{nk}}$$

$W_N^{2nk} = W_{\frac{N}{2}}^{k}$

In the above equation in the N/2 –point DFT of N/2 sequences is obtained by adding the first half and last half of the input sequences

When k is odd $e^{-j\pi k} = -1$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) - x_2(n)\right)W_N^{(2k+1)n}$$

$$\boxed{X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) - x_2(n)\right)W_{\frac{N}{2}}^{nk}\,W_N^{n}}$$

In the above equation in the N/2 –point DFT of N/2 sequences is obtained by subtracting the second half of the input from the first half and then multiplying the result with W_N^k

Decimation in Frequency (DIF)

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) + x_2(n)) W_{\frac{N}{2}}^{nk}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) - x_2(n)) W_{\frac{N}{2}}^{nk} W_N^n$$

Example : For N = 8

$x_1(n)$	$x_2(n)$
$x_1(0) = x(0)$	$x_2(0) = x(4)$
$x_1(1) = x(1)$	$x_2(1) = x(5)$
$x_1(2) = x(2)$	$x_2(2) = x(6)$
$x_1(3) = x(3)$	$x_2(3) = x(7)$

$$X(2k) = x_1(k) + x_2(k), \qquad for\ 0 \le k \le 3$$

$$X(2k+1) = [x_1(k) - x_2(k)]W_8^k, \qquad for\ 4 \le k \le 7$$

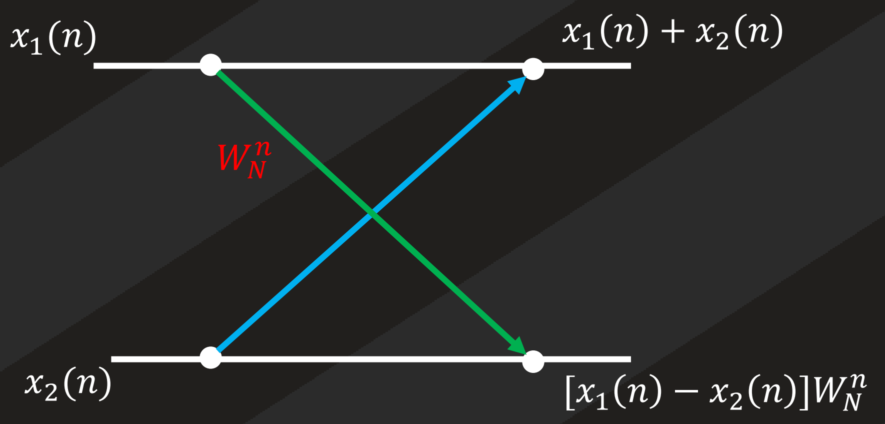
$X(0) = x_1(0) + x_2(0)$ $\qquad$ $X(4) = [x_1(0) - x_2(0)]W_8^0$

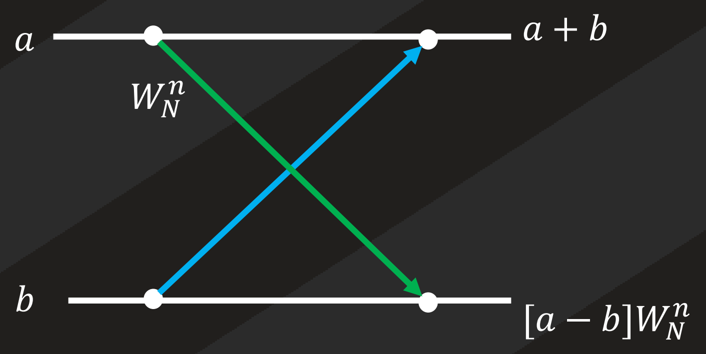
$X(1) = x_1(1) + x_2(1)$ $\qquad$ $X(5) = [x_1(1) - x_2(1)]W_8^1$

$X(2) = x_1(2) + x_2(2)$ $\qquad$ $X(6) = [x_1(2) - x_2(2)]W_8^2$

$X(3) = x_1(3) + x_2(3)$ $\qquad$ $X(7) = [x_1(3) - x_2(3)]W_8^3$

This operation can be represented by a butterfly diagram





Decimation in Frequency (DIF)

Steps to follow

Step 1 : Find the number of input samples (N)

Step 2 : input sequence in normal order

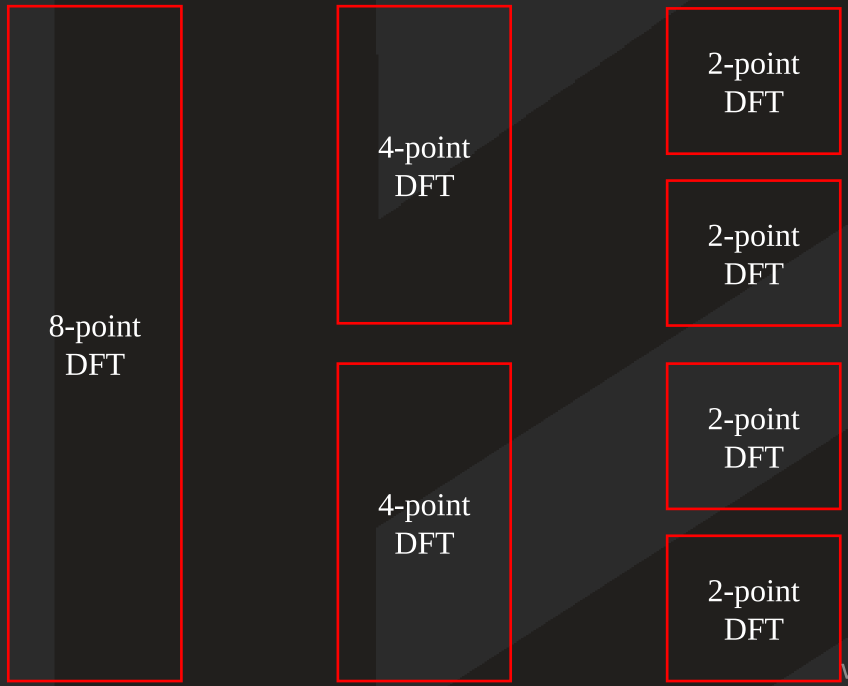
Step 3 : Calculate the number of stages ($M = \log_2 N$)

Step 4 : Calculate the number of max butterflies in stage (N/2)

Step 5 : Calculate the twiddle factor

Step 6 : Evaluate the N point DFT using butterfly diagram

Step7 : The DFT output is in bit-reversed order



Input	**Binary**	**Bit-reversed**	**Revised samples**
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Decimation in Frequency (DIF)

Q) Find the DFT of a sequence x(n) = {0, 1, 2, 3} using DIF algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : input sequence in normal order

Input
x(0)
x(1)
x(2)
x(3)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

Decimation in Frequency (DIF)

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^{M-m+1}} \qquad t = 0, 1, 2, \ldots 2^{M-m} - 1$$

Stage =1 $(M = 2, m = 1)$

$t = 0,1$

for t=0

$$k = \frac{Nt}{2^{M-m+1}} = \frac{4.0}{2^2} = 0$$

W_4^0 W_4^1

$$\boxed{W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1}$$

for t=1

$$k = \frac{4.1}{2^2} = 1$$

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).2} = e^{-j\pi} = \cos(\pi) - j\sin(\pi)$$

$$\boxed{W_4^1 = -j}$$

Stage =2 $(M = 2, m = 2)$

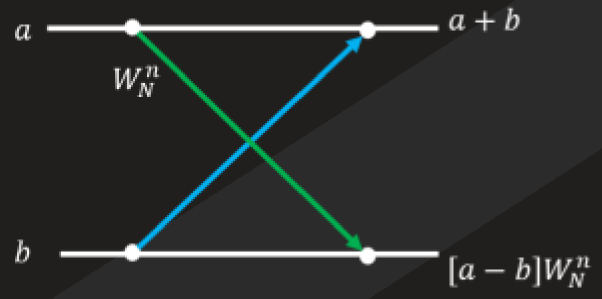
$t = 0$

$$k = \frac{Nt}{2^{M-m+1}} = \frac{4.0}{2^1} = 0$$

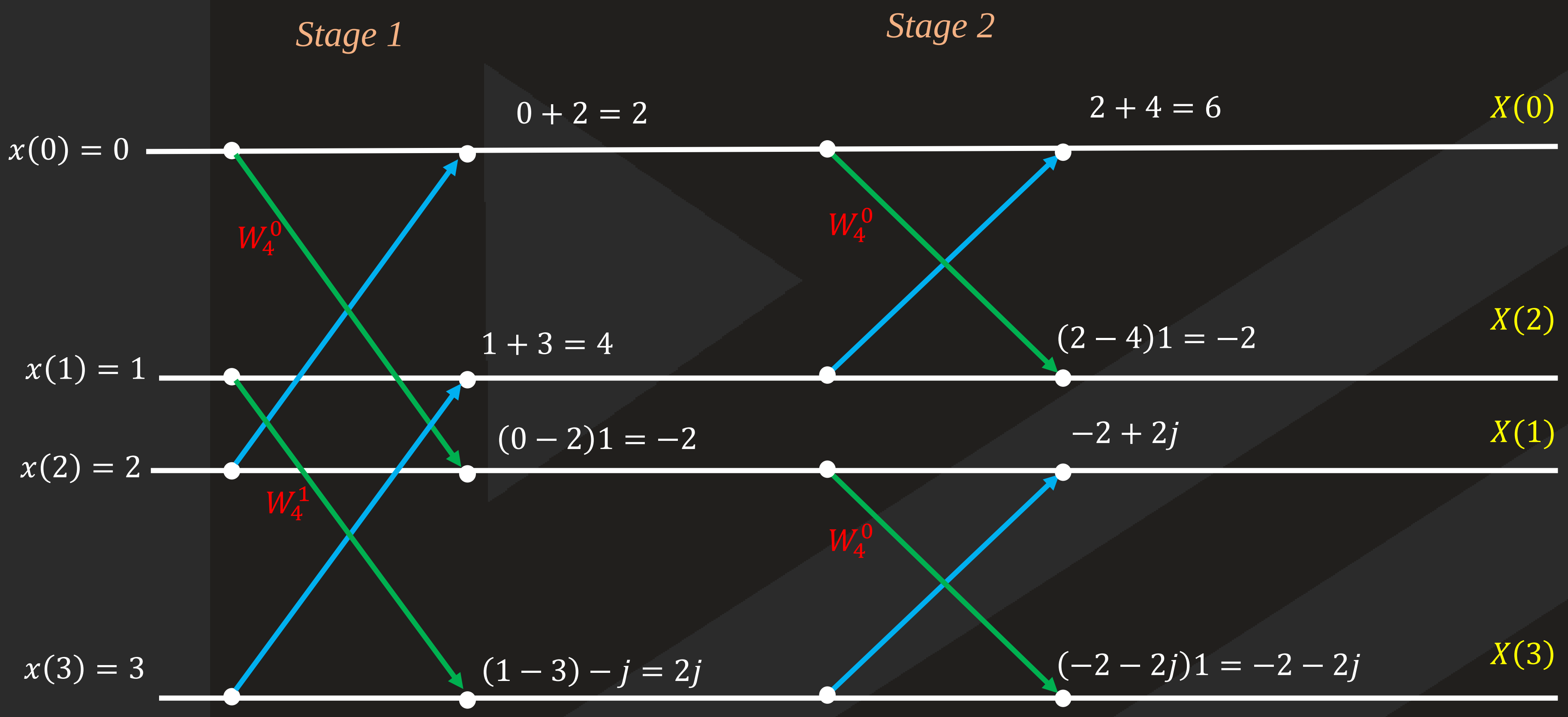
W_4^0

Decimation in Frequency (DIF)

Step 6 : Evaluate the N point DFT using butterfly diagram



$x(n) = \{0, 1, 2, 3\}$ $\quad$ $W_4^0 = 1$ $\quad$ $W_4^1 = -j$



Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step7 : The DFT output is in bit-reversed order

$X(k) = \{6, -2 + 2j, -2, -2 - 2j\}$

Decimation in Frequency (DIF)

Q) Find the DFT of a sequence x(n) = {1, 2, 3, 4, 4, 3, 2, 1 } using DIF algorithm

Solution

Step 1 : Find the number of input samples (N)

N=8

Step 2 : input sequence in normal order

Input	Binary	Bit-reversed	Revised samples
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 8$$

$$M = 3$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{8}{2} = 4$$

Decimation in Frequency (DIF)

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^{M-m+1}} \qquad t = 0, 1, 2, \dots 2^{M-m} - 1$$

Stage =1 $(M = 3, m = 1)$

$t = 0,1,2,3$

for t=0

$k = \frac{8.0}{2^3} = 0$ W_8^0 $\boxed{W_8^0 = 1}$

for t=1

$k = \frac{8.1}{2^3} = 1$ W_8^1 $\boxed{W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}$

for t=2

$k = \frac{8.2}{2^3} = 2$ W_8^2 $\boxed{W_8^2 = -j}$

for t=3

$k = \frac{8.3}{2^3} = 3$ W_8^3 $\boxed{W_8^2 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}$

Stage =2 $(M = 3, m = 2)$

$t = 0,1$

for t=0

$k = \frac{8.0}{2^2} = 0$ W_8^0 $\boxed{W_8^0 = 1}$

for t=1

$k = \frac{8.1}{2^2} = 2$ W_8^2 $\boxed{W_8^2 = -j}$

Stage =3 $(M = 3, m = 3)$

$t = 0$

for t=0

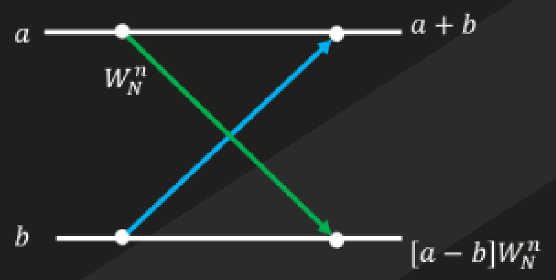
$k = \frac{8.0}{2^1} = 0$ W_8^0 $\boxed{W_8^0 = 1}$

$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.7071 - j0.7071$

$W_8^3 = -0.7071 - j0.7071$

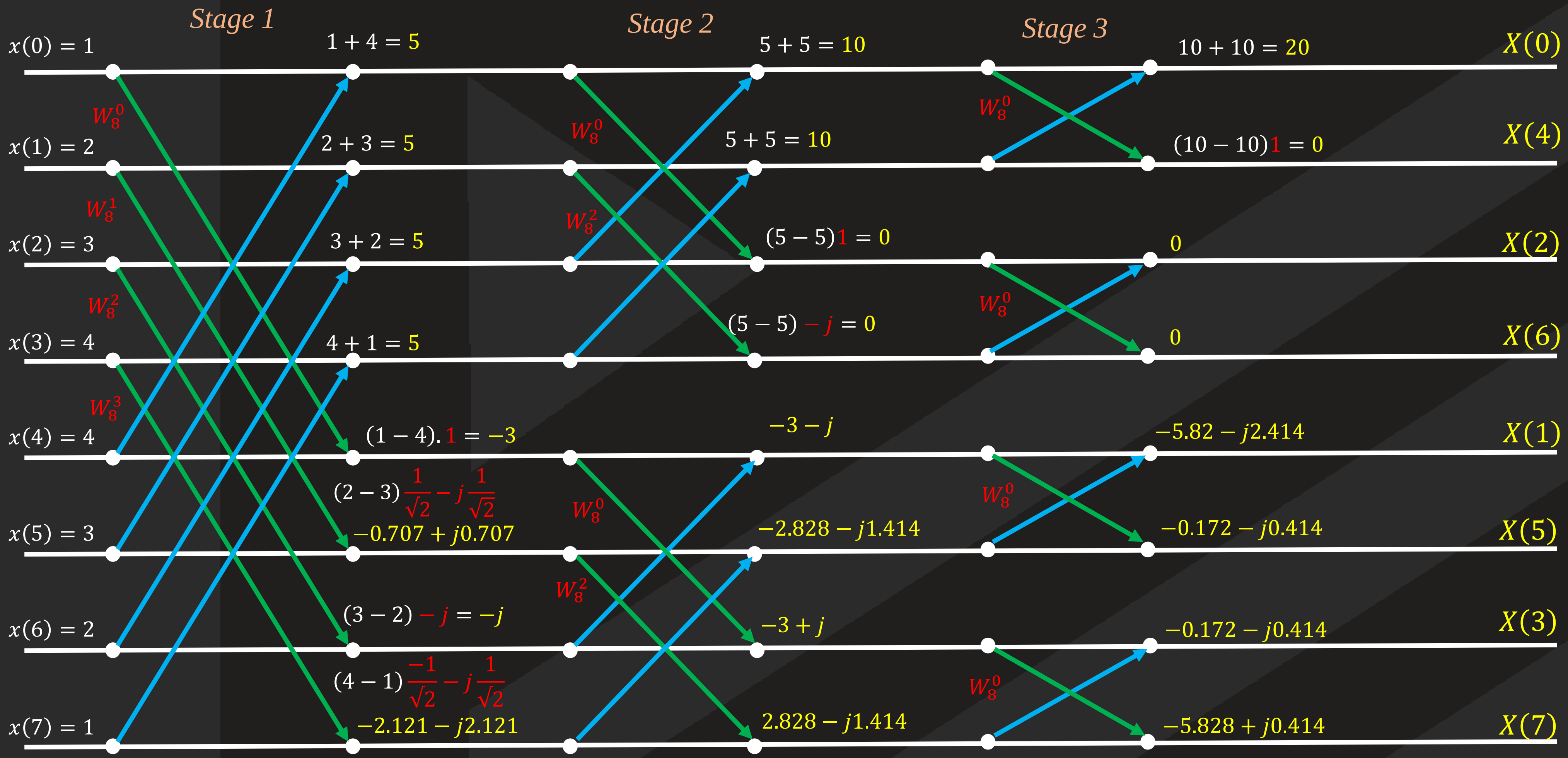
Decimation in Frequency (DIF)

Step 6 : Evaluate the N point DFT using butterfly diagram



$$x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$$

$W_8^0 = 1 \quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \quad W_8^2 = -j \quad W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$



Input	Revised samples
x(0)	x(0)
x(1)	x(4)
x(2)	x(2)
x(3)	x(6)
x(4)	x(1)
x(5)	x(5)
x(6)	x(3)
x(7)	x(7)

Step7 : The DFT output is in bit reversed order

$$X(k) = \{20, -5.82 - j2.414, 0, -0.172 - j0.414, 0, -0.172 - j0.414, 0, -5.828 + j0.414\}$$

IDFT Computation using Radix -2 FFT algorithm

The inverse DFT of an N-point sequence X(k), for k=0,1,2,…, N-1

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) W_N^{nk}\right]^*$$

Q) Find the IDFT of the sequence X(k) = {10, -2+2j, -2, -2-2j} using DIT algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : Bir reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

IDFT Computation using Radix -2 FFT algorithm

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^M} \qquad t = 0, 1, 2, \dots 2^{M-1} - 1$$

Stage =1 $\quad (M = 1)$

$$t = 0$$

$$k = \frac{Nt}{2^M} = \frac{4.0}{2^1} = 0$$

$$W_4^0$$

$$\boxed{W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1}$$

Stage =2 $\quad (M = 2)$

$$t = 0,1$$

for t=0

$$k = \frac{4.0}{2^2} = 0 \qquad W_4^0$$

for t=1

$$k = \frac{4.1}{2^2} = 1 \qquad W_4^1$$

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right)2} = e^{-j\pi} = \cos(\pi) - j\sin(\pi)$$

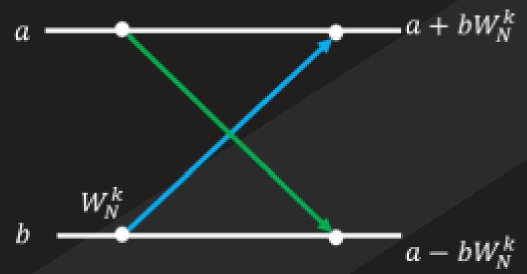
$$\boxed{W_4^1 = -j}$$

Step 6 : Find the conjugates of X(k)

X(k) = {10, -2+2j, -2, -2-2j}

$X^*(k)$ = {10, -2-2j, -2, -2+2j}

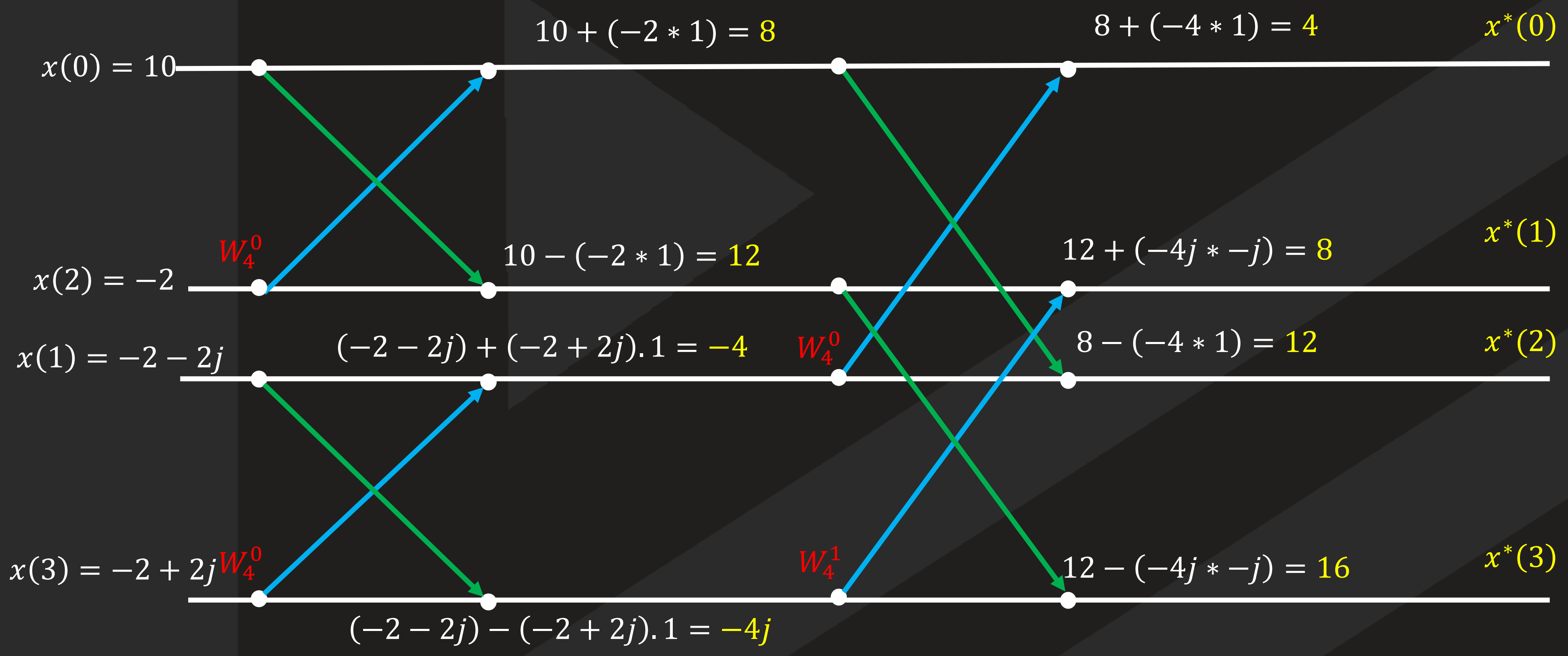
IDFT Computation using Radix -2 FFT algorithm



Step 7 : Evaluate the IDFT using butterfly diagram

$X^*(k) = \{10, -2-2j, -2, -2+2j\}$ $\quad W_4^0 = 1 \quad W_4^1 = -j$

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)



Step8 : The output is in normal order and divide it with N

$$x^*(n) = \frac{1}{4}\{4, 8, 12, 16\} \qquad x(n) = \{1, 2, 3, 4\}$$

IDFT Computation using Radix -2 FFT algorithm

Q) Find the IDFT of the sequence X(k) = {7, 2, 3, 1+j} using DIF algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : Bit reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step 3 : Calculate the number of stages ($M = \log_2 N$)

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

IDFT Computation using Radix -2 FFT algorithm

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^{M-m+1}} \qquad t = 0, 1, 2, \ldots 2^{M-m} - 1$$

Stage =1 $\quad (M = 2, m = 1)$

$t = 0,1$

for t=0

$$k = \frac{4.0}{2^2} = 0 \qquad W_4^0$$

for t=1

$$k = \frac{4.1}{2^2} = 1 \qquad W_4^1$$

$$\boxed{W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1}$$

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).2} \quad = e^{-j\pi} \qquad = \cos(\pi) - j\sin(\pi)$$

$$\boxed{W_4^1 = -j}$$

Stage =2 $\quad (M = 2, m = 2)$

$t = 0$

$$k = \frac{Nt}{2^M} \quad = \frac{4.0}{2^1} \quad = 0$$

W_4^0

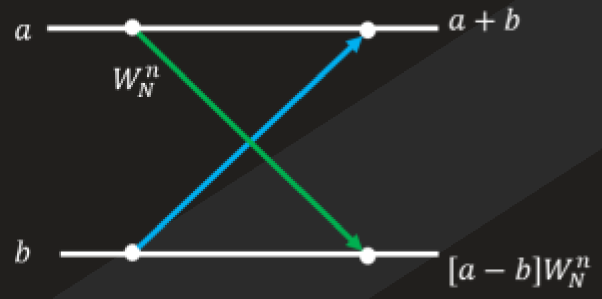
$$\boxed{W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1}$$

Step 6 : Find the conjugates of X(k)

$X(k) = \{7, 2, 3, 1+j\}$

$X^*(k) = \{7, 2, 3, 1-j\}$

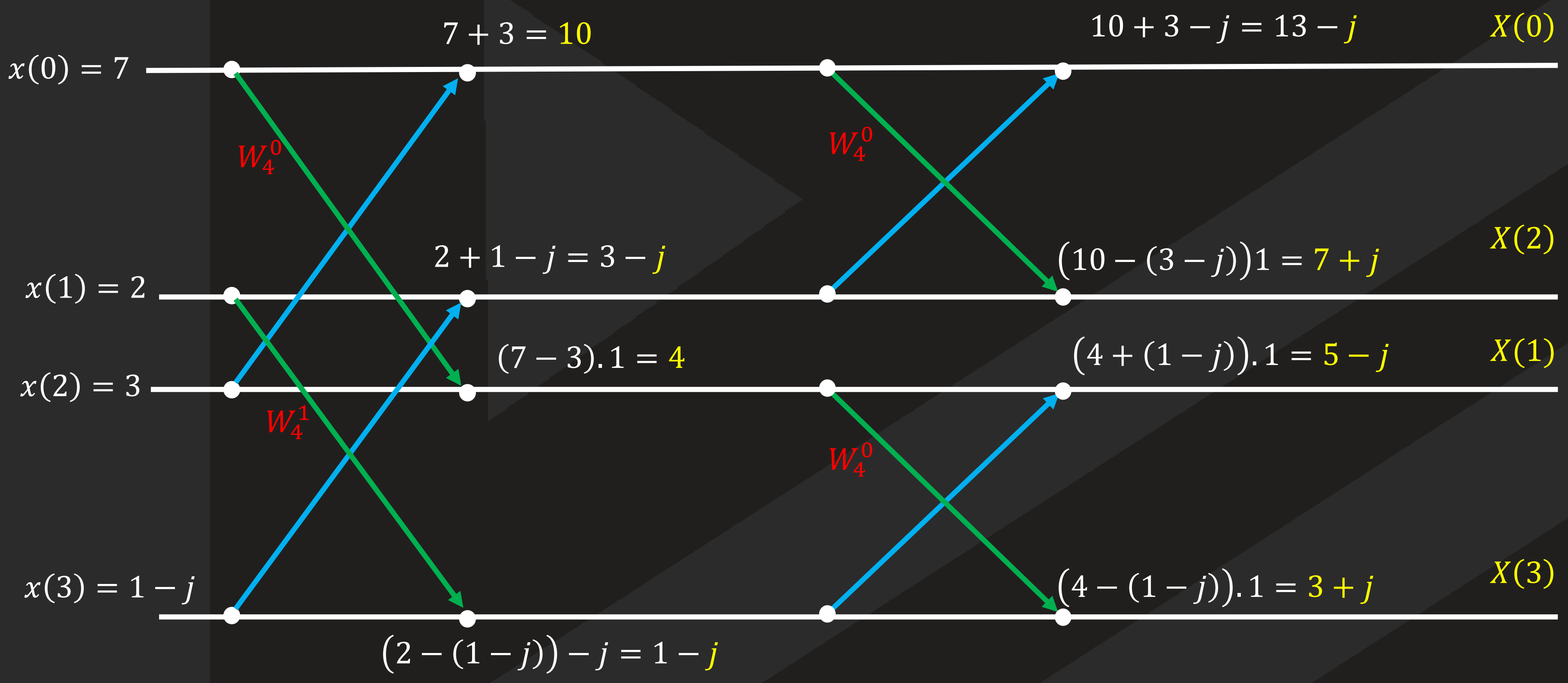
Decimation in Frequency (DIF)



Step 7 : Evaluate the N point DFT using butterfly diagram

$X^*(k) = \{7, 2, 3, 1-j\}$ $W_4^0 = 1$ $W_4^1 = -j$

Stage 1 *Stage 2*



Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step8 : The output is in normal order and divide it with N

$$x^*(n) = \frac{1}{4}\{13-j, 5-j, 7+j, 3+j\}$$

Application of FFT

Efficient computation of DFT of two real sequences

Let $x_1(n)$ and $x_2(n)$ are two real sequences of length N and let x(n) be a complex values sequence defined as $x(n) = x_1(n) + jx_2(n)$

Now find the DFT of the sequence x(n) which is linear

$$X(k) = X_1(k) + jX_2(k)$$

The sequences $x_1(n)$ and $x_2(n)$ can be expressed in terms of x(n) as

$$x_1(n) = \frac{x(n) + x^*(n)}{2} \qquad x_2(n) = \frac{x(n) - x^*(n)}{2j}$$

Then the DFT of $x_1(n)$ and $x_2(n)$ are

$$X_1(k) = \frac{1}{2}\{DFT[x(n)] + DFT[x^*(n)]\} \qquad X_2(k) = \frac{1}{2j}\{DFT[x(n)] - DFT[x^*(n)]\}$$

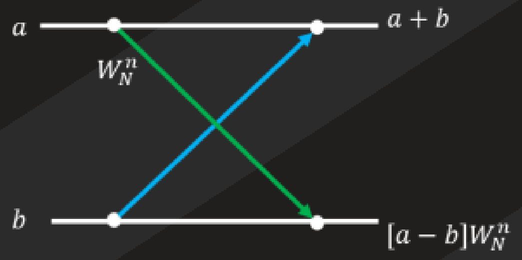
From conjugation property of twiddle factor

$$x^*(n) \overset{DFT}{\Longleftrightarrow} X^*(N-k)$$

$$\boxed{X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\} \qquad X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}}$$

Efficient computation of DFT of two real sequences

Q) Find the DFT of two sequence $x_1(n) = \{1,3,1,2\}$ and $x_2(n) = \{2,5,1,3\}$



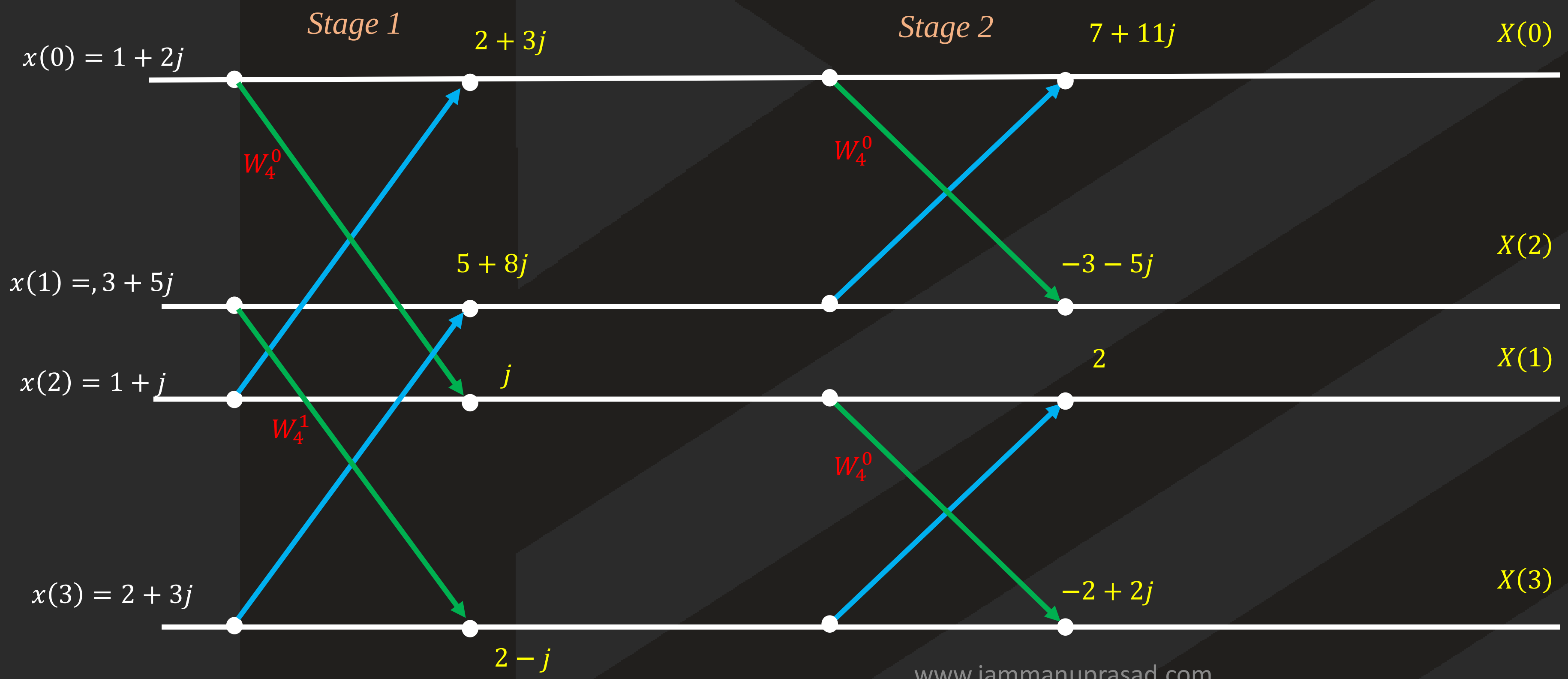
Solution

$$x(n) = x_1(n) + jx_2(n) \qquad \text{For n = 0,1,2,3} \qquad x(n) = \{1+2j, 3+5j, 1+j, 2+3j\}$$

Now find the DFT of the sequence x(n) using DIT or DIF method

$$x(n) = \{1+2j, 3+5j, 1+j, 2+3j\} \qquad W_4^0 = 1 \qquad W_4^1 = -j$$

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)



$$X(k) = \{7+11j, 2, -3-5j, -2+2j\}$$

$$X(k) = \{7+11j, 2, -3-5j, -2+2j\}$$

Now we have to calculate $X_1(k)$ and $X_2(k)$

For k=0

$$X_1(0) = \frac{1}{2}\{X(0) + X^*(4-0)\}$$

$$= \frac{1}{2}\{7+11j+7-11j\} \quad = 7$$

For k=1

$$X_1(1) = \frac{1}{2}\{X(1) + X^*(4-1)\}$$

$$= \frac{1}{2}\{2+(-2-2j)\} \quad = -j$$

For k=2

$$X_1(2) = \frac{1}{2}\{X(2) + X^*(4-2)\}$$

$$= \frac{1}{2}\{-3-5j+(-3+5j)\} \quad = -3$$

For k=3

$$X_1(3) = \frac{1}{2}\{X(3) + X^*(4-3)\}$$

$$= \frac{1}{2}\{-2+2j+(2)\} \quad = j$$

For k=0

$$X_2(0) = \frac{1}{2j}\{X(0) - X^*(4-0)\}$$

$$= \frac{1}{2j}\{7+11j-7-11j\} \quad = 11$$

For k=1

$$X_2(1) = \frac{1}{2j}\{X(1) - X^*(4-1)\}$$

$$= \frac{1}{2j}\{2-(-2-2j)\} \quad = -2j+1$$

For k=2

$$X_2(2) = \frac{1}{2j}\{X(2) - X^*(4-2)\}$$

$$= \frac{1}{2j}\{-3-5j-(-3+5j)\} \quad = -5$$

For k=3

$$X_2(3) = \frac{1}{2j}\{X(3) - X^*(4-3)\}$$

$$= \frac{1}{2}\{-2+2j-(2)\} \quad = 2j+1$$

$$X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\}$$

$$X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}$$

$$X_1(k) = \{7, -j, -3, j\}$$

$$X_2(k) = \{11, 1-2j, -5, 1+2j\}$$

Application of FFT

Efficient computation of DFT of a 2N-point real sequence

Let g(n) is a real valued sequences of 2N points.

To find the 2N point DFT from N point DFT , we divide the sequence to two

$$x_1(n) = g(2n) \qquad x_2(n) = g(2n+1)$$

Now follow same as the DFT computation of two real sequence

$$X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\} \qquad X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}$$

Finally we must express the 2N point DT in terms of two N point DFTs

$$G(k) = \sum_{n=0}^{N-1} g(2n)W_{2N}^{2nk} + \sum_{n=0}^{N-1} g(2n+1)W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} x_1(n)W_{2N}^{2nk} + W_{2N}^{k}\sum_{n=0}^{N-1} x_2(n)W_{2N}^{2nk}$$

$$W_N^2 = e^{\left(-\frac{j2\pi}{N}\right)2} = e^{\left(-\frac{j2\pi}{\frac{N}{2}}\right)} = W_{\frac{N}{2}}$$

$$W_{2N}^2 = e^{\left(-\frac{j2\pi}{2N}\right)2} = W_N$$

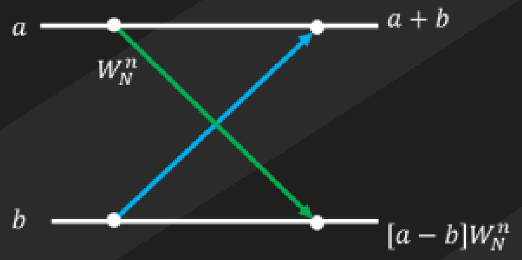
$$G(k) = \sum_{n=0}^{N-1} x_1(n)W_N^{nk} + W_{2N}^{k}\sum_{n=0}^{N-1} x_2(n)W_N^{nk}$$

$$\boxed{\begin{aligned} G(k) &= X_1(k) + W_{2N}^{k}X_2(k) \\ G(k+N) &= X_1(k) - W_{2N}^{k}X_2(k) \end{aligned}}$$

$$Where\ k = 0,1,2,\dots,N-1$$

Efficient computation of DFT of a 2N-point real sequence

Q) Find the DFT of the sequence x(n) = {1,3,7,2,1,2,1,3} using 4-point DFT



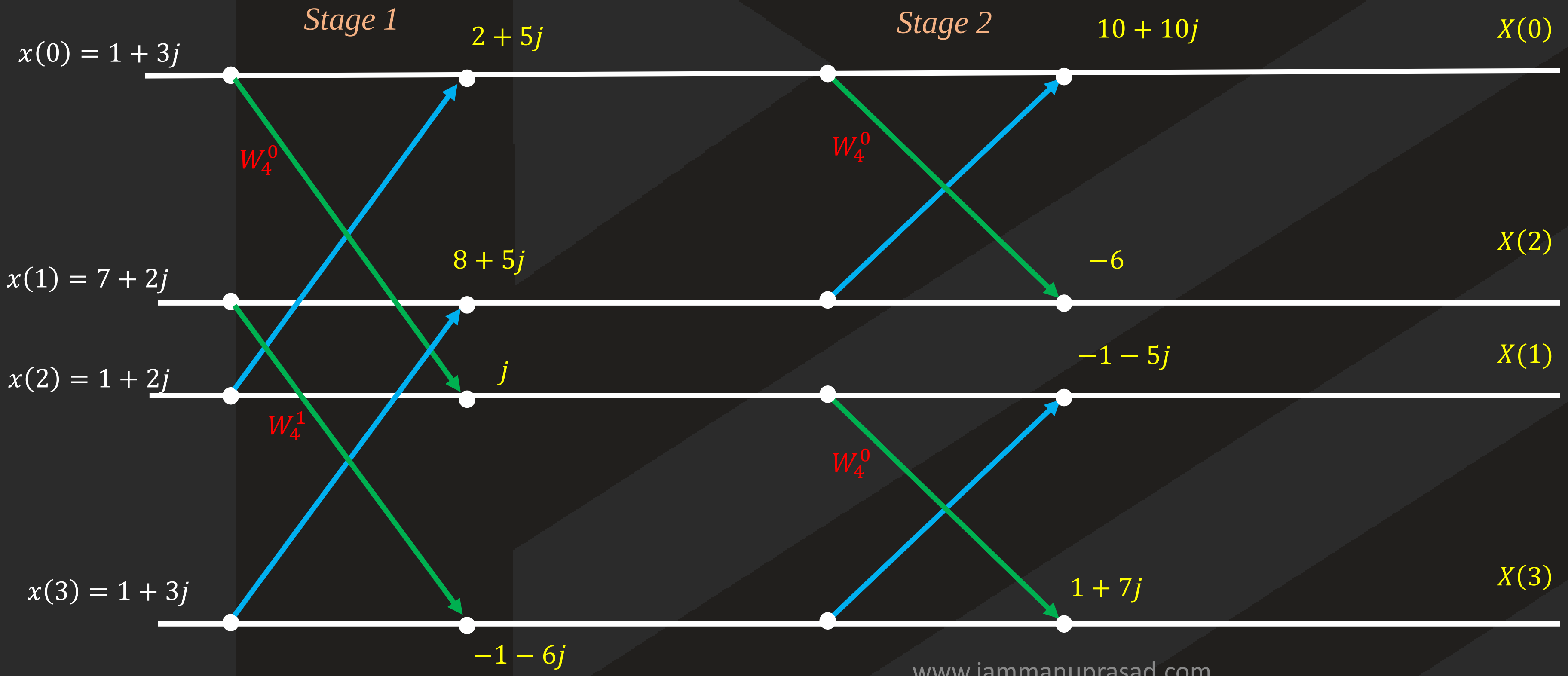
Solution $\quad x_1(n) = \{1,7,1,1,\} \qquad x_2(n) = \{3,2,2,3\}$

$$x(n) = x_1(n) + jx_2(n) \qquad\qquad x(n) = \{1+3j, 7+2j, 1+2j, 1+3j\}$$

Now find the DFT of the sequence x(n) using DIT or DIF method

$$x(n) = \{1+3j, 7+2j, 1+2j, 1+3j\} \qquad W_4^0 = 1 \qquad W_4^1 = -j$$

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)



$$X(k) = \{10+10j, -1-5j, -6, 1+7j\}$$

$$X(k) = \{10 + 10j, -1 - 5j, -6, 1 + 7j\}$$

$$X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\}$$

$$X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}$$

Now we have to calculate $X_1(k)$ and $X_2(k)$

For k=0

$$X_1(0) = \frac{1}{2}\{X(0) + X^*(4-0)\}$$

$$= \frac{1}{2}\{10 + 10j + 10 - 10j\} \quad = 10$$

For k=1

$$X_1(1) = \frac{1}{2}\{X(1) + X^*(4-1)\}$$

$$= \frac{1}{2}\{-1 - 5j + 1 - 7j\} = -6j$$

For k=2

$$X_1(2) = \frac{1}{2}\{X(2) + X^*(4-2)\}$$

$$= \frac{1}{2}\{-6 + (-6)\} \quad = -6$$

For k=3

$$X_1(3) = \frac{1}{2}\{X(3) + X^*(4-3)\}$$

$$= \frac{1}{2}\{1 + 7j + (-1 + 5j)\} \quad = 6j$$

For k=0

$$X_2(0) = \frac{1}{2j}\{X(0) - X^*(4-0)\}$$

$$= \frac{1}{2j}\{10+10j-(10-10j)\} = 10$$

For k=1

$$X_2(1) = \frac{1}{2j}\{X(1) + X^*(4-1)\}$$

$$= \frac{1}{2j}\{-1 - 5j - (1 - 7j)\} \quad = 1 + j$$

For k=2

$$X_2(2) = \frac{1}{2j}\{X(2) + X^*(4-2)\}$$

$$= \frac{1}{2j}\{-6 - (-6)\} \quad = 0$$

For k=3

$$X_2(3) = \frac{1}{2j}\{X(3) - X^*(4-3)\}$$

$$= \frac{1}{2}\{1 + 7j - (-1 + 5j)\} \quad = 1 - j$$

$$X_1(k) = \{10, -6j, -6, 6j\}$$

$$X_2(k) = \{10, 1 + j, 0, 1 - j\}$$

$X_1(k) = \{10, -6j, -6, 6j\}$

$X_2(k) = \{10, 1+j, 0, 1-j\}$

Here 2N =8 so,

$W_8^0 = 1 \quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \quad W_8^2 = -j \quad W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

For k=0

$X(0) = X_1(0) + W_8^0 X_2(0)$

$= 10 + 1.[10] \quad = 20$

For k=1

$X(1) = X_1(1) + W_8^1 X_2(1)$

$= -6j + [1+j]\left[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = \sqrt{2} - 6j$

For k=2

$X(2) = X_1(2) + W_8^2 X_2(2)$

$= -6 + 0[-j] \quad = -6$

For k=3

$X(3) = X_1(3) + W_8^3 X_2(3)$

$= 6j + [1-j]\left[\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = -\sqrt{2} + 6j$

$\boxed{\begin{aligned} G(k) &= X_1(k) + W_{2N}^k X_2(k) \\ G(k+N) &= X_1(k) - W_{2N}^k X_2(k) \end{aligned}}$

For k=0

$X(0+4) = X_1(0) - W_8^0 X_2(0)$

$= 10 - 1.[10] \quad = 0$

For k=1

$X(1+4) = X_1(1) - W_8^1 X_2(1)$

$= -6j - [1+j]\left[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = -\sqrt{2} - 6j$

For k=2

$X(2+4) = X_1(2) - W_8^2 X_2(2)$

$= -6 - 0[-j] \quad = -6$

For k=3

$X(3+4) = X_1(3) - W_8^3 X_2(3)$

$= 6j - [1-j]\left[\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = \sqrt{2} + 6j$

$X(k) = \{20, \sqrt{2} - 6j, -6, -\sqrt{2} + 6j, 0, -\sqrt{2} - 6j, -6, \sqrt{2} + 6j\}$